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General Relativity

We now have all of the concepts in hand to state the content of GR.

In doing so we need to address 2 points:

a) How does spacetime get curved?

b) Once we know the geometry of spacetime, how does this affect the behavior of particles?

a) Einstein's Equation(s) $$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\equiv G_{\mu\nu} \text{ the Einstein tensor}} = 8\pi G\, T_{\mu\nu}$$

symmetric under $\mu \leftrightarrow \nu$ so

10 independent eqns. in 4D

All sources (mass, energy, pressure, E+M fields, etc.) are in $T_{\mu\nu}$.

We then solve for $g_{\mu\nu}(x^\mu)$

symmetric under $\mu \leftrightarrow \nu$ so 10 ind. comp. in 4D

10 equations and 10 unknowns sounds perfect!

But $R^\mu{}_{\nu\rho\lambda}$ also satisfies a geometric condition called the

This means it does not come from varying an action (dynamics) but is instead a built in feature of what we use to construct the action.

Branchi identity: $\nabla^\mu R_{\rho\mu} = \frac{1}{2} \nabla_\rho R$ 4 differential relations since $\rho = 0,1,2,3$

Before we mentioned index symmetries of $R^\mu{}_{\nu\rho\lambda}$, but these are differential.

These four expressions can be used to identify that 4 of the 10 expressions in Einstein's equations are not independent.

In simple terms suppose we have 3 differential equations for 3 unknown functions:

$E_1(f,g,h)$, $E_2(f,g,h)$, $E_3(f,g,h)$

but then suppose they also satisfied the identity $\underbrace{E_1 + E_2 + E_3 = 0}$ $\Rightarrow E_1 = -(E_2 + E_3)$

Note this must be a differential equation as well.

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It's not so bad and you've seen this before:

$$\vec{\nabla}\cdot\vec{E} = \rho$$
$$\vec{\nabla}\times\vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}$$

} 4 equations for 6 unknowns

These are the dynamical content of E+M

subject to:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0$$

} 4 more equations.

These are the "geometric" conditions

8 equations for 6 unknowns sounds bad, but the last two suggest $\vec{E} = \vec{\nabla}\phi$, $\vec{B} = \vec{\nabla}\times\vec{A}$ leaving 4 unknowns $(\phi, \vec{A})$ which can be found from the first two equations.

But there is a bit of a subtlety here since we can combine the first two equations by considering $\vec{\nabla}\cdot\vec{\nabla}\times\vec{B} = \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = 0$

using the charge-current continuity identity $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

then gives $-\frac{\partial \rho}{\partial t} + \frac{\partial(\vec{\nabla}\cdot\vec{E})}{\partial t} = 0$

or $\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E} - \rho) = 0$

This tells us that if we set $\vec{\nabla}\cdot\vec{E} = \rho$ initially it is guaranteed for all time, i.e. it is non-dynamical. This means that $\phi, \vec{A}$ is actually underconstrained! But we should expect that since E&M enjoys gauge invariance, i.e. $\phi \to \phi + \partial_t \lambda$

$\vec{A} \to \vec{A} + \vec{\nabla}\lambda$

In GR the Bianchi identity actually reduces us to a system of 6 equations for 10 unknowns, but this is what we should expect! Solving for $g_{\mu\nu}(x^\mu)$ we know that we can always do a coordinate change $x^\mu \to x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} x^\mu$ which also changes $g_{\mu\nu} \to g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$. These 4 coordinate redefinitions do not change the underlying geometry, so they should represent the same solution.

In summary: For a given $T_{\mu\nu}$ there is one unique geometry that can be represented by a family of metrics $g_{\mu\nu}$ related to each other by coordinate transformations.

b) Minimal Coupling Principle

Once we have a curved spacetime, we can figure out how our familiar laws of physics get modified using the following recipe:

1. Start with a law valid in an inertial frame in flat space.
2. Write the law in terms of true 4D tensors.
3. Assert that the tensor form is true in curved space as well.

In practice these mean:

1. Start with a Lorentz invariant theory in terms of tensors.
2. Replace $\underbrace{\eta_{\mu\nu}}_{\text{flat space metric}} \rightarrow \underbrace{g_{\mu\nu}}_{\text{curved space metric}}$ and $\underbrace{\partial_\mu}_{\text{a good derivative in flat space}} \rightarrow \underbrace{\nabla_\mu}_{\text{a good derivative in curved space}}$ everywhere.

These steps allow us to generalize <u>any</u> flat space theory to curved spacetime.

Example: E+M

Tensorial Maxwell's equations in flat spacetime: $\partial_\mu F^{\mu\nu} = J^\nu$

$$\partial_{[\mu} F_{\nu\lambda]} = 0$$

where $F_{\nu\lambda} = \eta_{\nu\alpha}\eta_{\lambda\beta}F^{\alpha\beta}$

$$\Downarrow$$

Maxwell's equations in curved spacetime: $\nabla_\mu F^{\mu\nu} = J^\nu$

$$\nabla_{[\mu} F_{\nu\lambda]} = 0$$

where $F_{\nu\lambda} = g_{\nu\alpha} g_{\lambda\beta} F^{\alpha\beta}$

Example: Gravity

In flat spacetime there is no gravity so particles move along straight lines:

$$x^\mu(\lambda) = a^\mu \lambda + x_0^\mu$$

or $$\frac{d^2 x^\mu}{d\lambda^2} = 0$$

or $$\frac{dx^\nu}{d\lambda} \partial_\nu \frac{dx^\mu}{d\lambda} = 0 \qquad \left(\text{using } \frac{d}{d\lambda} = \frac{dx^\nu}{d\lambda}\partial_\nu\right)$$

Then in curved spacetime: $$\frac{dx^\nu}{d\lambda} \nabla_\nu \frac{dx^\mu}{d\lambda} = 0$$

or $$\frac{dx^\nu}{d\lambda}\left(\partial_\nu + \Gamma^\mu_{\ \nu\lambda}\right) \frac{dx^\lambda}{d\lambda} = 0$$

or $$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\ \nu\lambda} \frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda} = 0$$ The geodesic equation!

Now that we have the basic content of GR down, we should recall that this theory is supposed to generalize (be a relativistic version of) Newtonian gravity.

So where is Newton?

Recall the 2 parts of GR:

a) How does space get curved? Einstein's equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

b) How does curved space influence motion? Geodesic Equation $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$

These each have a non-relativistic limit in the Newtonian theory:

a) $\nabla^2 \phi = 4\pi G \rho$ where ϕ is the gravitational potential (hence $\vec{g} = -\vec{\nabla}\phi$) and ρ is mass density

b) $\vec{a} = -\vec{\nabla}\phi$ which is just Newton's second law w/ gravity after cancelling masses.

We will extract each of these Newtonian results from the corresponding equations in GR using the following limits:

a) Small velocities $\frac{dx^i}{dt} << c = 1$ or $\frac{dx^i}{d\tau} << \frac{dt}{d\tau} = \frac{dx^0}{d\tau}$ (recall i is a spatial index)

b) Weak grav. field $\underbrace{g_{\mu\nu}}_{\text{full metric}} = \underbrace{\eta_{\mu\nu}}_{\text{flat metric}} + \underbrace{h_{\mu\nu}}_{\text{small perturbation}}$ w/ $\|h_{\mu\nu}\| << 1$ (so ignore terms $\mathcal{O}(h^2)$ and higher)

c) Static grav. field $\partial_0 g_{\mu\nu} = 0$ (let's us avoid gravito-magnetic effects)

Due to b, we will need an inverse metric which is: $g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}$

To see that this is the right choice consider: $g_{\lambda\mu} g^{\mu\nu} = \delta_\lambda{}^\nu$

$$= (\eta_{\lambda\mu} + h_{\lambda\mu})(\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta})$$

$$= \eta_{\lambda\mu}\eta^{\mu\nu} - \eta_{\lambda\mu}\eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta} + \eta^{\mu\nu}h_{\lambda\mu} + \mathcal{O}(h^2)$$

$$= \delta_\lambda{}^\nu - \delta_\lambda{}^\alpha \eta^{\nu\beta} h_{\alpha\beta} + \eta^{\mu\nu} h_{\lambda\mu}$$

$$= \delta_\lambda{}^\nu - \eta^{\nu\beta} h_{\lambda\beta} + \eta^{\mu\nu} h_{\lambda\mu}$$

$$= \delta_\lambda{}^\nu \quad \checkmark$$

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Let's start by looking for $\vec{a} = -\vec{\nabla}\phi$ in $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$.

Consider a massive particle and let $\lambda = \tau$ (proper time). Then:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dx^0}{d\tau}\right)^2 \qquad \text{using (a)}$$

$$\text{but } \Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}\Big(\underbrace{\underset{0}{\cancel{\partial_0} g_{\lambda 0}} + \underset{0}{\cancel{\partial_0} g_{0\lambda}}}_{\text{by (c)}} - \underbrace{\partial_\lambda g_{00}}_{= -\partial_\lambda h_{00} \text{ by (b)}}\Big) = -\frac{1}{2}(\eta^{\mu\lambda} - \eta^{\mu\alpha}\eta^{\lambda\beta}h_{\alpha\beta})\partial_\lambda h_{00} \simeq -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} \qquad \text{using (b)}$$

Then:

$$\frac{d^2x^\mu}{d\tau^2} - \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \qquad \text{4 equations (one for each } \mu)$$

$$\mu = 0 \qquad \frac{d^2t}{d\tau^2} - \frac{1}{2}\underbrace{\eta^{0\lambda}}_{\text{only } \eta^{00} = -1 \text{ is nonzero}}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 = \frac{d^2t}{d\tau^2} + \frac{1}{2}\underset{0 \text{ by (c)}}{\cancel{\partial_0 h_{00}}}\left(\frac{dt}{d\tau}\right)^2$$

Hence $\frac{d^2t}{d\tau^2} = 0$ which means we can take $t = \tau$ which is good because we know that in nonrelativistic physics we can adequately parameterize motion w/ coordinate time t.

$$\mu = i \qquad \frac{d^2x^i}{dt^2} - \frac{1}{2}\underbrace{\eta^{i\lambda}}_{\text{only } \eta^{ii} = 1 \text{ is nonzero}}\partial_\lambda h_{00}\left(\frac{dt}{dt}\right)^2 = 0 = a^i - \frac{1}{2}\partial_i h_{00} \Rightarrow a^i = -\partial_i\big(\underbrace{-\tfrac{1}{2}h_{00}}\big)$$

If we identify this as $\phi = -\frac{1}{2}h_{00}$

then $\boxed{\vec{a} = -\vec{\nabla}\phi}$

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Now we look for $\nabla^2\phi = 4\pi G\rho$ in $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$.

First we rewrite Einstein's eqn. in trace-reverse form.

Trace over both sides:

$$g^{\mu\nu}(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) = g^{\mu\nu}(8\pi G T_{\mu\nu})$$

$$R - 2R = 8\pi G T$$

$$R = -8\pi G T$$

Then: $$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}(-8\pi G T) = 8\pi G T_{\mu\nu}$$

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T) \quad \text{Trace-reverse form of EE}$$

We need to specify a source. In Newtonian gravity only mass contributes to gravity (not energy, momentum, pressure, etc.) so we take: $T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$ (Perfect fluid in its rest frame)

Then: $$T = g^{\mu\nu}T_{\mu\nu} = (\eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta})T_{\mu\nu} \quad \text{using (b)}$$

$$= (\eta^{00} - \underbrace{\eta^{0\alpha}\eta^{0\beta}}_{}h_{\alpha\beta})T_{00} \quad \text{since only } T_{00} = \rho \neq 0$$

only $\eta^{00} = -1$ is nonzero

$$= (-1 - h_{00})\rho$$

So the r.h.s. of trace-reverse EE becomes:

$$8\pi G(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T) = 8\pi G[T_{00} - \tfrac{1}{2}g_{00}(-1-h_{00})\rho]$$

$$= 8\pi G[\rho - \tfrac{1}{2}(-1+h_{00})(-1-h_{00})\rho] \qquad g_{00} = \underbrace{\eta_{00}}_{-1} + h_{00}$$

$$= 8\pi G[\rho - \tfrac{1}{2}\rho + O(h^2)]$$

$$= 4\pi G\rho$$

For the l.h.s. we have:

$$R_{00} = R^{\lambda}{}_{0\lambda 0} = \partial_\lambda \Gamma^\lambda_{00} - \overset{=0 \text{ by (c)}}{\cancel{\partial_0\Gamma^\lambda_{\lambda 0}}} + \overbrace{\overset{0}{\cancel{\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{00}}} - \overset{0}{\cancel{\Gamma^\lambda_{0\rho}\Gamma^\rho_{\lambda 0}}}}^{O(h^2)}$$

$$= -\underset{0 \text{ by (c)}}{\cancel{\partial_0\Gamma^0_{00}}} + \partial_i \underbrace{\Gamma^i_{00}}_{\frac{1}{2}g^{i\lambda}(\overbrace{\cancel{\partial_0 g_{\lambda 0}} + \cancel{\partial_0 g_{0\lambda}}}^{0 \text{ by (c)}} - \partial_\lambda g_{00})}$$

$\uparrow$ only $\lambda \neq 0$ will contribute by (c)

$$= -\tfrac{1}{2}\partial_i g^{ij}\partial_j g_{00}$$

$$= -\tfrac{1}{2}\partial_i\partial^i(-1 + h_{00}) \quad \text{since } g_{00} = \underbrace{\eta_{00}}_{-1} + h_{00}$$

$$= -\tfrac{1}{2}\nabla^2(h_{00}) \quad \text{since } \partial_i\partial^i(-1) = 0$$

$$= \nabla^2\phi \quad \text{using } \phi = -\tfrac{1}{2}h_{00} \text{ \underline{as we did before}!!}$$

Then: $\nabla^2\phi = 4\pi G\rho$ BAM!!